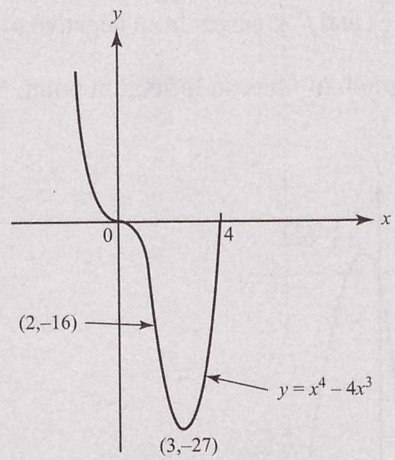


FIGURE N4–5

Verify the preceding on your calculator.

CASE II. FUNCTIONS WHOSE DERIVATIVES MAY NOT EXIST EVERYWHERE.

If there are values of x for which a first or second derivative does not exist, we consider those values separately, recalling that a local maximum or minimum point is one of transition between intervals of rise and fall and that an inflection point is one of transition between intervals of upward and downward concavity.

EXAMPLE 14

Sketch the graph of $y = x^{2/3}$.

SOLUTION: $\frac{dy}{dx} = \frac{2}{3x^{1/3}}$ and $\frac{d^2y}{dx^2} = -\frac{2}{9x^{4/3}}$.

Neither derivative is zero anywhere; both derivatives fail to exist when $x = 0$. As x increases through 0, $\frac{dy}{dx}$ changes from – to +; (0, 0) is therefore a minimum. Note that the tangent is vertical at the origin, and that since $\frac{d^2y}{dx^2}$ is negative everywhere except at 0, the curve is everywhere concave down. See Figure N4–6.

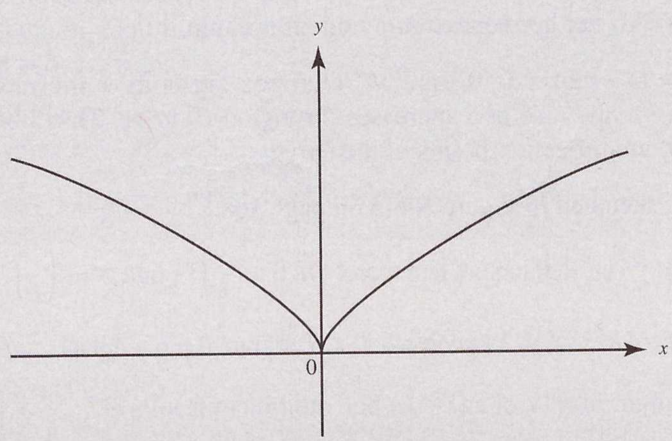


FIGURE N4–6

EXAMPLE 15

Sketch the graph of $y = x^{1/3}$.

SOLUTION: $\frac{dy}{dx} = \frac{1}{3x^{2/3}}$ and $\frac{d^2y}{dx^2} = -\frac{2}{9x^{5/3}}$.

As in Example 14, neither derivative ever equals zero and both fail to exist when $x = 0$. Here, however, as x increases through 0, $\frac{dy}{dx}$ does not change sign. Since $\frac{dy}{dx}$ is positive for all x except 0, the curve rises for all x and can have neither maximum nor minimum points. The tangent is again vertical at the origin. Note here that $\frac{d^2y}{dx^2}$ does change sign (from + to –) as x increases through 0, so that (0, 0) is a point of inflection of the curve. See Figure N4–7.

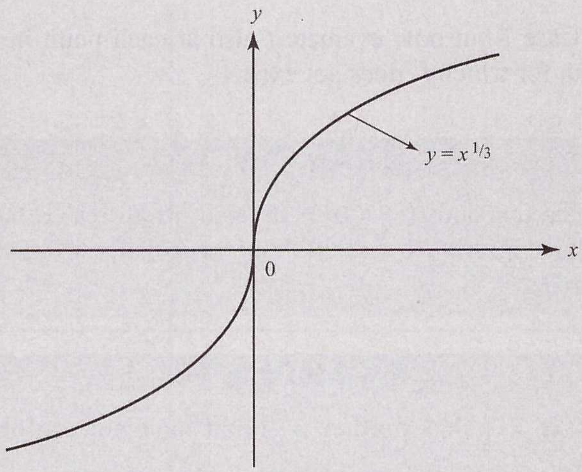


FIGURE N4–7

Verify the graph on your calculator.

F. GLOBAL MAXIMUM OR MINIMUM

CASE I. DIFFERENTIABLE FUNCTIONS.

If a function f is differentiable on a closed interval $a \leqslant x \leqslant b$, then f is also continuous on the closed interval $[a,b]$ and we know from the Extreme Value Theorem (page 101) that f attains both a (global) maximum and a (global) minimum on $[a,b]$. To find these, we solve the equation $f'(x) = 0$ for critical points on the interval $[a,b]$, then evaluate f at each of those and also at $x = a$ and $x = b$. The largest value of f obtained is the global max, and the smallest the global min.

EXAMPLE 16

Find the global max and global min of f on (a) $-2 \leqslant x \leqslant 3$, and (b) $0 \leqslant x \leqslant 3$, if $f(x) = 2x^3 - 3x^2 - 12x$.

SOLUTION:

(a) $f'(x) = 6x^2 - 6x - 12 = 6(x + 1)(x - 2)$, which equals zero if $x = -1$ or 2. Since $f(-2) = -4$, $f(-1) = 7$, $f(2) = -20$, and $f(3) = -9$, the global max of f occurs at $x = -1$ and equals 7, and the global min of f occurs at $x = 2$ and equals -20.

(b) Only the critical value 2 lies in [0,3]. We now evaluate f at 0, 2, and 3. Since $f(0) = 0$, $f(2) = -20$, and $f(3) = -9$, the global max of f equals 0 and the global min equals -20.

CASE II. FUNCTIONS THAT ARE NOT EVERYWHERE DIFFERENTIABLE.

We proceed as for Case I but now evaluate f also at each point in a given interval for which f is defined but for which f' does not exist.

EXAMPLE 17

The absolute-value function $f(x) = |x|$ is defined for all real x, but $f'(x)$ does not exist at $x = 0$. Since $f'(x) = -1$ if $x < 0$, but $f'(x) = 1$ if $x > 0$, we see that f has a global min at $x = 0$.

EXAMPLE 18

The function $f(x) = \frac{1}{x}$ has neither a global max nor a global min on *any* interval that contains zero (see Figure N2–4, page 90). However, it does attain both a global max and a global min on every closed interval that does not contain zero. For instance, on [2,5] the global max of f is $\frac{1}{2}$, the global min $\frac{1}{5}$.

G. FURTHER AIDS IN SKETCHING

It is often very helpful to investigate one or more of the following before sketching the graph of a function or of an equation:

(1) Intercepts. Set $x = 0$ and $y = 0$ to find any y- and x-intercepts respectively.

(2) Symmetry. Let the point (x, y) satisfy an equation. Then its graph is symmetric about
- the x-axis if $(x, -y)$ also satisfies the equation;
- the y-axis if $(-x, y)$ also satisfies the equation;
- the origin if $(-x, -y)$ also satisfies the equation.

(3) Asymptotes. The line $y = b$ is a horizontal asymptote of the graph of a function f if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$. If $f(x) = \frac{P(x)}{Q(x)}$, inspect the degrees of $P(x)$ and $Q(x)$, then use the Rational Function Theorem, page 96. The line $x = c$ is a vertical asymptote of the rational function $\frac{P(x)}{Q(x)}$ if $Q(c) = 0$ but $P(c) \neq 0$.

(4) Points of discontinuity. Identify points not in the domain of a function, particularly where the denominator equals zero.

EXAMPLE 19

Sketch the graph of $y = \frac{2x+1}{x-1}$.

SOLUTION: If $x = 0$, then $y = -1$. Also, $y = 0$ when the numerator equals zero, which is when $x = -\frac{1}{2}$. A check shows that the graph does not possess any of the symmetries described above. Since $y \to 2$ as $x \to \pm\infty$, $y = 2$ is a horizontal asymptote; also, $x = 1$ is a vertical asymptote. The function is defined for all reals except $x = 1$; the latter is the only point of discontinuity.

We find derivatives: $y' = -\frac{3}{(x-1)^2}$ and $y'' = \frac{6}{(x-1)^3}$.

From y' we see that the function decreases everywhere (except at $x = 1$), and from y'' that the curve is concave down if $x < 1$, up if $x > 1$. See Figure N4–8.

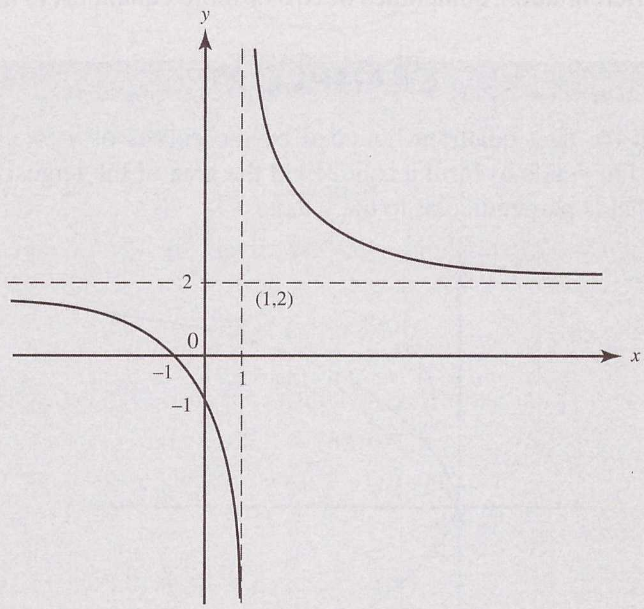


FIGURE N4–8

Verify the preceding on your calculator, using $[-4,4] \times [-4,8]$.

EXAMPLE 20

Describe any symmetries of the graphs of

(a) $3y^2 + x = 2$; (b) $y = x + \frac{1}{x}$; (c) $x^2 - 3y^2 = 27$.

SOLUTIONS:

(a) Suppose point (x, y) is on this graph. Then so is point $(x, -y)$, since $3(-y)^2 + x = 2$ is equivalent to $3y^2 + x = 2$. Then (a) is symmetric about the x-axis.

(b) Note that point $(-x, -y)$ satisfies the equation if point (x, y) does:

$$(-y) = (-x) + \frac{1}{(-x)} \leftrightarrow y = x + \frac{1}{x}.$$

Therefore the graph of this function is symmetric about the origin.

(c) This graph is symmetric about the x-axis, the y-axis, and the origin. It is easy to see that, if point (x, y) satisfies the equation, so do points $(x, -y)$, $(-x, y)$, and $(-x, -y)$.

H. OPTIMIZATION: PROBLEMS INVOLVING MAXIMA AND MINIMA

The techniques described above can be applied to problems in which a function is to be maximized (or minimized). Often it helps to draw a figure. If y, the quantity to be maximized (or minimized), can be expressed explicitly in terms of x, then the procedure outlined above can be used. If the domain of y is restricted to some closed interval, one should always check the endpoints of this interval so as not to overlook possible extrema. Often, implicit differentiation, sometimes of two or more equations, is indicated.

EXAMPLE 21

The region in the first quadrant bounded by the curves of $y^2 = x$ and $y = x$ is rotated about the y-axis to form a solid. Find the area of the largest cross section of this solid that is perpendicular to the y-axis.

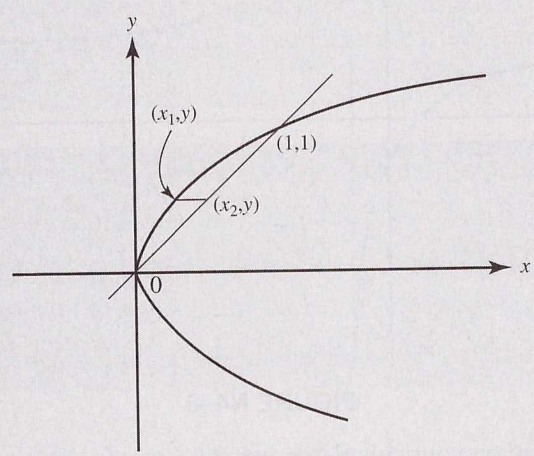


FIGURE N4–9

SOLUTION: See Figure N4–9. The curves intersect at the origin and at (1,1), so $0 < y < 1$. A cross section of the solid is a ring whose area A is the difference between the areas of two circles, one with radius x_2, the other with radius x_1. Thus

$$A = \pi x_2^{\,2} - \pi x_1^{\,2} = \pi(y^2 - y^4);\quad \frac{dA}{dy} = \pi(2y - 4y^3) = 2\pi y(1 - 2y^2).$$

The only relevant zero of the first derivative is $y = \frac{1}{\sqrt{2}}$. There the area A is

$$A = \pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{4}.$$

Note that $\frac{d^2A}{dy^2} = \pi(2 - 12y^2)$ and that this is negative when $y = \frac{1}{\sqrt{2}}$, assuring a maximum there. Note further that A equals zero at each endpoint of the interval [0,1] so that $\frac{\pi}{4}$ is the global maximum area.

EXAMPLE 22

The volume of a cylinder equals V cubic inches, where V is a constant. Find the proportions of the cylinder that minimize the total surface area.

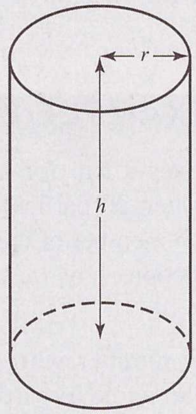


FIGURE N4–10

SOLUTION: We know that the volume is

$$V = \pi r^2 h \tag{1}$$

where r is the radius and h the height. We seek to minimize S, the total surface area, where

$$S = 2\pi r^2 + 2\pi rh \tag{2}$$

Solving (1) for h, we have $h = \dfrac{V}{\pi r^2}$, which we substitute in (2):

$$S = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}. \tag{3}$$

Differentiating (3) with respect to r yields

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2}.$$

Now we set $\dfrac{dS}{dr}$ equal to zero to determine the conditions that make S a minimum:

$$4\pi r - \frac{2V}{r^2} = 0$$
$$4\pi r = \frac{2V}{r^2}$$
$$4\pi r = \frac{2\left(\pi r^2 h\right)}{r^2}$$
$$2r = h.$$

The total surface area of a cylinder of fixed volume is thus a minimum when its height equals its diameter.

(Note that we need not concern ourselves with the possibility that the value of r that renders $\frac{dS}{dr}$ equal to zero will produce a maximum surface area rather than a minimum one. With V fixed, we can choose r and h so as to make S as large as we like.)

EXAMPLE 23

A charter bus company advertises a trip for a group as follows: At least 20 people must sign up. The cost when 20 participate is \$80 per person. The price will drop by \$2 per ticket for each member of the traveling group in excess of 20. If the bus can accommodate 28 people, how many participants will maximize the company's revenue?

SOLUTION: Let x denote the number who sign up in excess of 20. Then $0 \leqq x \leqq 8$. The total number who agree to participate is $(20 + x)$, and the price per ticket is $(80 - 2x)$ dollars. Then the revenue R, in dollars, is

$$\begin{aligned} R &= (20 + x)(80 - 2x), \\ R'(x) &= (20 + x)(-2) + (80 - 2x) \cdot 1 \\ &= 40 - 4x. \end{aligned}$$

$R'(x)$ is zero if $x = 10$. Although $x = 10$ yields maximum R—note that $R''(x) = -4$ and is always negative—this value of x is not within the restricted interval. We therefore evaluate R at the endpoints 0 and 8: $R(0) = 1600$ and $R(8) = 28 \cdot 64 = 1792$, 28 participants will maximize revenue.

EXAMPLE 24

A utilities company wants to deliver gas from a source S to a plant P located across a straight river 3 miles wide, then downstream 5 miles, as shown in Figure N4–11. It costs \$4 per foot to lay the pipe in the river but only \$2 per foot to lay it on land.

(a) Express the cost of laying the pipe in terms of u.

(b) How can the pipe be laid most economically?

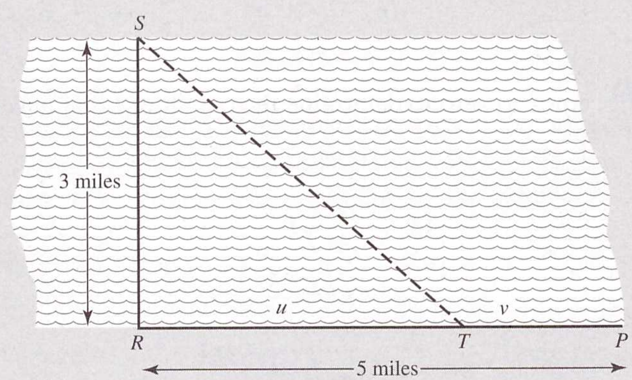


FIGURE N4–11

SOLUTIONS:

(a) Note that the problem "allows" us to (1) lay all of the pipe in the river, along a line from S to P; (2) lay pipe along SR, in the river, then along RP on land; or (3) lay some pipe in the river, say, along ST, and lay the rest on land along TP. When T coincides with P, we have case (1), with $v = 0$; when T coincides with R, we have case (2), with $u = 0$. Case (3) includes both (1) and (2).

In any event, we need to find the lengths of pipe needed (that is, the distances involved); then we must figure out the cost.

In terms of u:

	In the River	On Land
Distances:		
miles	$ST = \sqrt{9+u^2}$	$TP = v = 5 - u$
feet	$ST = 5280\sqrt{9+u^2}$	$TP = 5280(5 - u)$
Costs (dollars):	$4(5280)\sqrt{9+u^2}$	$2[5280(5 - u)]$

If $C(u)$ is the total cost,

$$C(u) = 21{,}120\sqrt{9+u^2} + 10{,}560(5-u)$$
$$= 10{,}560(2\sqrt{9+u^2} + 5 - u).$$

(b) We now minimize $C(u)$:

$$C'(u) = 10{,}560\left(2\cdot\frac{1}{2}\frac{2u}{\sqrt{9+u^2}} - 1\right) = 10{,}560\left(\frac{2u}{\sqrt{9+u^2}} - 1\right).$$

We now set $C'(u)$ equal to zero and solve for u:

$$\frac{2u}{\sqrt{9+u^2}} - 1 = 0 \;\rightarrow\; \frac{2u}{\sqrt{9+u^2}} = 1 \;\rightarrow\; \frac{4u^2}{9+u^2} = 1,$$

where, in the last step, we squared both sides; then

$$4u^2 = 9 + u^2, \qquad 3u^2 = 9, \qquad u^2 = 3, \qquad u = \sqrt{3},$$

where we discard $u = -\sqrt{3}$ as meaningless for this problem.

The domain of $C(u)$ is $[0,5]$ and C is continuous on $[0,5]$. Since

$$C(0) = 10{,}560\left(2\sqrt{9} + 5\right) = \$116{,}160,$$
$$C(5) = 10{,}560\left(2\sqrt{34}\right) \approx \$123{,}150,$$
$$C\left(\sqrt{3}\right) = 10{,}560\left(2\sqrt{12} + 5 - \sqrt{3}\right) = \$107{,}671,$$

So $u = \sqrt{3}$ yields minimum cost. Thus, the pipe can be laid most economically if some of it is laid in the river from the source S to a point T that is $\sqrt{3}$ miles toward the plant P from R, and the rest is laid along the road from T to P.

I. RELATING A FUNCTION AND ITS DERIVATIVES GRAPHICALLY

The following table shows the characteristics of a function f and their implications for f's derivatives. These are crucial in obtaining one graph from another. The table can be used reading from left to right or from right to left.

Note that the slope at $x = c$ of any graph of a function is equal to the ordinate at c of the derivative of the function.

<table>
<tr><th></th><th>f</th><th colspan="3">f'</th><th colspan="3">f''</th></tr>
<tr><td>ON AN INTERVAL</td><td>increasing
decreasing</td><td colspan="3">≥ 0
≤ 0</td><td colspan="3"></td></tr>
<tr><td rowspan="4">AT c</td><td></td><td>$x < c$</td><td>$x = c$</td><td>$x > c$</td><td colspan="3"></td></tr>
<tr><td>local maximum</td><td>+</td><td>0</td><td>–</td><td colspan="3" rowspan="1">$f''(c) < 0$</td></tr>
<tr><td></td><td colspan="3">(f' is decreasing)</td><td colspan="3"></td></tr>
<tr><td>local minimum</td><td>–
(f' is increasing)</td><td>0</td><td>+</td><td colspan="3">$f''(c) > 0$</td></tr>
<tr><td></td><td>neither local maximum nor local minimum</td><td>+
–</td><td>0
0</td><td>+
–</td><td colspan="3"></td></tr>
<tr><td></td><td></td><td colspan="3">(f' does not change sign)</td><td colspan="3"></td></tr>
<tr><td rowspan="3">AT c</td><td></td><td colspan="3"></td><td>$x < c$</td><td>$x = c$</td><td>$x > c$</td></tr>
<tr><td rowspan="2">point of inflection</td><td colspan="3">$f'(c)$ is a minimum;
f' changes from decreasing to increasing</td><td>–</td><td>0</td><td>+</td></tr>
<tr><td colspan="3">$f'(c)$ is a maximum;
f' changes from increasing to decreasing</td><td>+</td><td>0</td><td>–</td></tr>
<tr><td rowspan="2">ON AN INTERVAL</td><td>concave up</td><td colspan="3">f' is increasing</td><td colspan="3">$f'' \geq 0$</td></tr>
<tr><td>concave down</td><td colspan="3">f' is decreasing</td><td colspan="3">$f'' \leq 0$</td></tr>
</table>

If $f'(c)$ does not exist, check the signs of f' as x increases through c: plus-to-minus yields a local maximum; minus-to-plus yields a local minimum; no sign change means no maximum or minimum, but check the possibility of a point of inflection.

AN IMPORTANT NOTE:

Tables and number lines showing sign changes of the function and its derivatives can be very helpful in organizing all of this information. *Note, however, that the AP Exam requires that students write sentences that describe the behavior of the function based on the sign of its derivative.*

EXAMPLE 25A

Given the graph of $f(x)$ shown in Figure N4–12, sketch $f'(x)$.

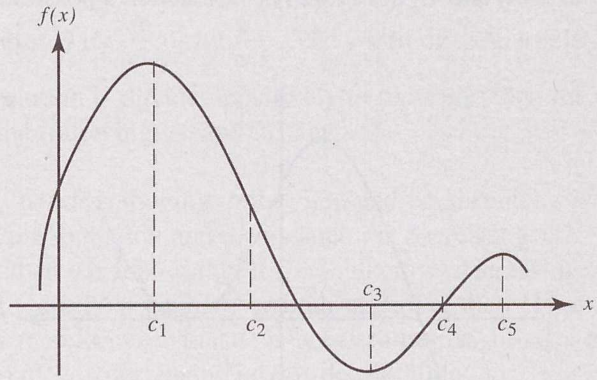
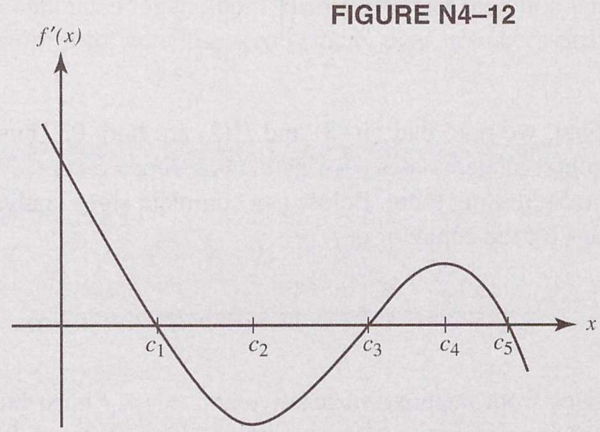


FIGURE N4–12



Point $x =$	Behavior of f	Behavior of f'
c_1	$f(c_1)$ is a local max	$f'(c_1) = 0$; f' changes sign from + to –
c_2	c_2 is an inflection point of f; the graph of f changes concavity from down to up	f' changes from decreasing to increasing; $f'(c_2)$ is a local minimum
c_3	$f(c_3)$ is a local minimum	$f'(c_3) = 0$; f' changes sign from – to +
c_4	c_4 is an inflection point of f; the graph of f changes concavity from up to down	f' changes from increasing to decreasing; $f'(c_4)$ is a local maximum
c_5	$f(c_5)$ is a local maximum	$f'(c_5) = 0$; f' changes sign from + to –

EXAMPLE 25B

Given the graph of $f'(x)$ shown in Figure N4–13, sketch a possible graph of f.

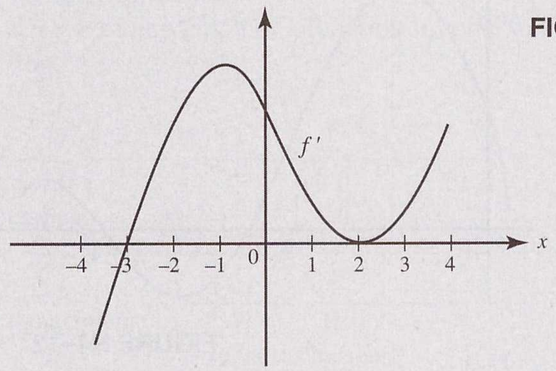


FIGURE N4–13

SOLUTION: First, we note that $f'(-3)$ and $f'(2)$ are both 0. Thus the graph of f must have horizontal tangents at $x = -3$ and $x = 2$. Since $f'(x) < 0$ for $x < -3$, we see that f must be decreasing there. Below is a complete signs analysis of f', showing what it implies for the behavior of f.

f	dec	-3	inc	2	inc
f'	$-$		$+$		$+$

Because f' changes from negative to positive at $x = -3$, f must have a minimum there, but f has neither a minimum nor a maximum at $x = 2$.
We note next from the graph that f' is increasing for $x < -1$. This means that the derivative of f', f'', must be positive for $x < -1$ and that f is concave upward there. Analyzing the signs of f'' yields the following:

f	conc. upward	-1	conc. down	2	conc. upward
f'	inc		dec		inc
f''	$+$		$-$		$+$

We conclude that the graph of f has two points of inflection, because it changes concavity from upward to downward at $x = -1$ and back to upward at $x = 2$. We use the information obtained to sketch a possible graph of f, shown in Figure N4–14. Note that other graphs are possible; in fact, any vertical translation of this f will do!

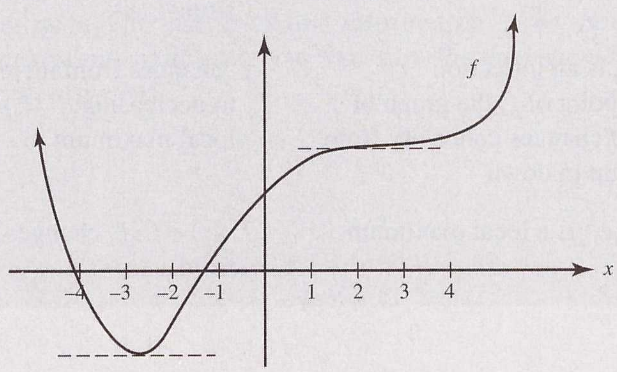


FIGURE N4–14

J. MOTION ALONG A LINE

Velocity

Acceleration

Speed

If a particle moves along a line according to the law $s = f(t)$, where s represents the position of the particle P on the line at time t, then the velocity v of P at time t is given by $\frac{ds}{dt}$ and its acceleration a by $\frac{dv}{dt}$ or by $\frac{d^2s}{dt^2}$. The speed of the particle is $|v|$, the magnitude of v. If the line of motion is directed positively to the right, then the motion of the particle P is subject to the following: At any instant,

(1) if $v > 0$, then P is moving to the right and its distance s is increasing; if $v < 0$, then P is moving to the left and its distance s is decreasing;
(2) if $a > 0$, then v is increasing; if $a < 0$, then v is decreasing;
(3) if a and v are both positive or both negative, then (1) and (2) imply that the speed of P is increasing or that P is accelerating; if a and v have opposite signs, then the speed of P is decreasing or P is decelerating;
(4) if s is a continuous function of t, then P reverses direction whenever v is zero and a is different from zero; note that zero velocity does not necessarily imply a reversal in direction.

EXAMPLE 26

A particle moves along a line such that its position $s = 2t^3 - 9t^2 + 12t - 4$, for $t \geqq 0$.
(a) Find all t for which the distance s is increasing.
(b) Find all t for which the velocity is increasing.
(c) Find all t for which the speed of the particle is increasing.
(d) Find the speed when $t = \frac{3}{2}$.
(e) Find the total distance traveled between $t = 0$ and $t = 4$.

SOLUTION: $v = \dfrac{ds}{dt} = 6t^2 - 18t + 12 = 6(t^2 - 3t + 2) = 6(t-2)(t-1)$

and $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = 12t - 18 = 12\left(t - \dfrac{3}{2}\right).$

Velocity $v = 0$ at $t = 1$ and $t = 2$, and:

if	$t < 1,$	then	$v > 0,$
	$1 < t < 2,$		$v < 0,$
	$t > 2,$		$v > 0.$

Acceleration $a = 0$ at $t = \frac{3}{2}$, and:

if	$t < \frac{3}{2},$	then	$a < 0,$
	$t > \frac{3}{2},$		$a > 0.$

These signs of v and a immediately yield the answers, as follows:

(a) s increases when $t < 1$ or $t > 2$.

(b) v increases when $t > \frac{3}{2}$.

(c) The speed $|v|$ is increasing when v and a are both positive, that is, for $t > 2$, and when v and a are both negative, that is, for $1 < t < \frac{3}{2}$.

(d) The speed when $t = \frac{3}{2}$ equals $|v| = \left|-\frac{3}{2}\right| = \frac{3}{2}$.

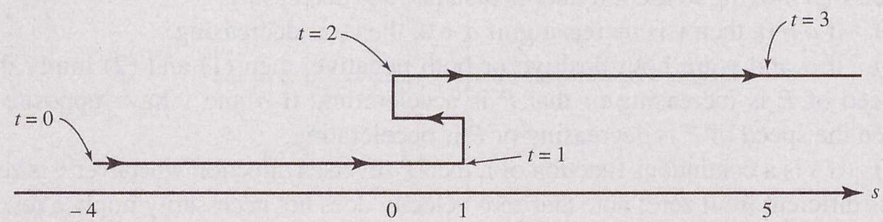


FIGURE N4–15

(e) P's motion can be indicated as shown in Figure N4–15. P moves to the right if $t < 1$, reverses its direction at $t = 1$, moves to the left when $1 < t < 2$, reverses again at $t = 2$, and continues to the right for all $t > 2$. The position of P at certain times t are shown in the following table:

t:	0	1	2	4
s:	−4	1	0	28

Thus P travels a total of 34 units between times $t = 0$ and $t = 4$.

EXAMPLE 27

Answer the questions of Example 26 if the law of motion is

$$s = t^4 - 4t^3.$$

SOLUTION: Since $v = 4t^3 - 12t^2 = 4t^2(t - 3)$ and $a = 12t^2 - 24t = 12t(t - 2)$, the signs of v and a are as follows:

if	$t < 3$,	then	$v < 0$
	$3 < t$,		$v > 0$;
if	$t < 0$,	then	$a > 0$
	$0 < t < 2$,		$a < 0$
	$2 < t$,		$a > 0$.

Thus

(a) s increases if $t > 3$.

(b) v increases if $t < 0$ or $t > 2$.

(c) Since v and a have the same sign if $0 < t < 2$ or if $t > 3$, the speed increases on these intervals.

(d) The speed when $t = \frac{3}{2}$ equals $|v| = \left|-\frac{27}{2}\right| = \frac{27}{2}$.

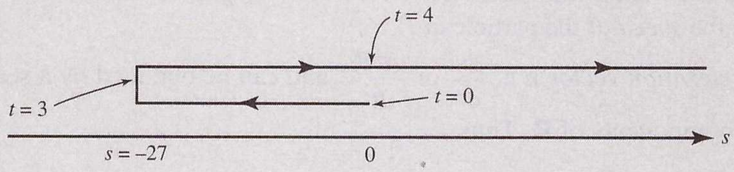


FIGURE N4–16

(e) The motion is shown in Figure N4–16. The particle moves to the left if $t < 3$ and to the right if $t > 3$, stopping instantaneously when $t = 0$ and $t = 3$, but reversing direction only when $t = 3$. Thus:

t:	0	3	4
s:	0	–27	0

The particle travels a total of 54 units between $t = 0$ and $t = 4$.

(Compare with Example 13, page 167, where the function $f(x) = x^4 - 4x^3$ is investigated for maximum and minimum values; also see the accompanying Figure N4–5 on page 168.)

K. MOTION ALONG A CURVE: VELOCITY AND ACCELERATION VECTORS

BC ONLY

Vector

Components

If a point P moves along a curve defined parametrically by $P(t) = (x(t), y(t))$, where t represents time, then the vector from the origin to P is called the *position vector*, with x as its *horizontal component* and y as its *vertical component*. The set of position vectors for all values of t in the domain common to $x(t)$ and $y(t)$ is called *vector function*.

A vector may be symbolized either by a boldface letter (**R**) or an italic letter with an arrow written over it ($\vec{R}$). The position vector, then, may be written as $\vec{R}(t) = \langle x,y \rangle$ or as $\mathbf{R} = \langle x,y \rangle$. In print the boldface notation is clearer, and will be used in this book; when writing by hand, the arrow notation is simpler.

Velocity vector

The *velocity vector* is the derivative of the vector function (the position vector):

$$\mathbf{v} = \frac{d\mathbf{R}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \quad \text{or} \quad \vec{v}(t) = \left\langle \frac{dx}{dy}, \frac{dy}{dt} \right\rangle.$$

Alternative notations for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are v_x and v_y, respectively; these are the components of **v** in the horizontal and vertical directions, respectively. The slope of **v** is

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx},$$

which is the slope of the curve; the *magnitude* of **v** is the vector's length:

Magnitude

$$|\mathbf{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{v_x^{\,2} + v_y^{\,2}}.$$